Two-band random matrices

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Spectral correlations in unitary invariant, non-Gaussian ensembles of large random matrices possessing an eigenvalue gap are studied within the framework of the orthogonal polynomial technique. Both local and global characteristics of spectra are directly reconstructed from the recurrence equation for orthogonal polynomials associated with a given random matrix ensemble. It is established that an eigenvalue gap does not affect the local eigenvalue correlations that follow the universal sine and the universal multicritical laws in the bulk and soft-edge scaling limits, respectively. By contrast, global smoothed eigenvalue correlations do reflect the presence of a gap, and are shown to satisfy a new universal law exhibiting a sharp dependence on the odd or even dimension of random matrices whose spectra are bounded. In the case of an unbounded spectrum, the corresponding universal ''density-density'' correlator is conjectured to be generic for chaotic systems with a forbidden gap and broken time reversal symmetry. [S1063-651X(98)04206-8]

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I. INTRODUCTION

Ensembles of large random matrices $\mathbf{H}$ generated by the joint distribution function $P[\mathbf{H}]\propto \exp\{-\beta \operatorname{Tr} V[\mathbf{H}]\}$, with β being a symmetry parameter as explained below, may display phase transitions under nonmonotonic deformation of the confinement potential $V[\mathbf{H}]$. Different phases are characterized by topologically different arrangements of eigenvalues in random matrix spectra that may have multiple-band structure. Random matrices, whose spectra undergo phase transitions, appear in quantizing two-dimensional gravity [1–3], in the context of quantum chromodynamics [4,5], as well as in some models of particles interacting in high dimensions [6]. Transition regimes realized in invariant random matrix ensembles have implications for a certain class of Calogero-Sutherland-Moser models [7]. These matrix models may also be applicable to chaotic systems having a forbidden gap in the energy spectrum.

In the eigenvalue representation, the invariant random matrix model is defined by the joint probability distribution function [8]

$$P(\{\varepsilon\})=\mathcal{Z}_N^{-1}\prod_{i>j=1}^{N}|\varepsilon_i-\varepsilon_j|^{\beta}\prod_{k=1}^{N}\exp\{-\beta V(\varepsilon_k)\} \quad (1)$$

of N eigenvalues $\{\varepsilon\}=\{\varepsilon_1,\ldots,\varepsilon_N\}$ of an $N\times N$ random matrix $\mathbf{H}$. The symmetry parameter β coincides with a number of independent elements in off-diagonal entries of a random matrix $\mathbf{H}$. For real symmetric matrices, $\beta=1$ (orthogonal symmetry), $\beta=2$ for Hermitian matrices (unitary symmetry), and $\beta=4$ for self-dual Hermitian matrices (symplectic symmetry). It is convenient to parametrize the confinement potential $V(\varepsilon)$ entering Eq. (1) by a set of coupling constants $\{d\}=\{d_1,\ldots,d_p\}$,

$$V(\varepsilon)=\sum_{k=1}^{p}\frac{d_k}{2k}\varepsilon^{2k},\quad d_p>0, \quad (2)$$

so that we may consider the phase transitions as occurring in $\{d\}$ space. Because the confinement potential is an even function, the associated random matrix model possesses so-called $Z2$ symmetry.

Variations of the coupling constants affect the Dyson density ν_D, that can be found by minimizing the free energy $\mathcal{F}_N=-\ln\mathcal{Z}_N$, Eq. (1), subject to a normalization constraint $\int \nu_D(\varepsilon)d\varepsilon=N$,

$$\frac{dV}{d\varepsilon}-\mathrm{P}\int d\zeta\,\frac{\nu_D(\zeta)}{\varepsilon-\zeta}=0, \quad (3)$$

where P indicates a principal value of the integral. When all d_k are positive, so that the confinement potential is monotonic, the spectral density ν_D has a single-band support, $\mathcal{N}_b=1$. Nonmonotonic deformation of the confinement potential can be carried out by changing the signs of some of d_k $(k\neq p)$. Such a *continuous* variation of coupling constants may lead, under certain conditions, to a *discontinuous* change of the topological structure of spectral density ν_D, when the eigenvalues $\{\varepsilon\}$ are arranged in $\mathcal{N}_b>1$ ''allowed'' bands separated by ''forbidden'' gaps.

The phase structure of the Hermitian $(\beta=2)$ one-matrix model Eq. (1) has been studied in a number of works [9–12], where the simplest examples of nonmonotonic quartic and sextic confinement potentials have been examined. It has been found that there are domains in the phase space of coupling constants where only a particular solution for ν_D exists, and it has a fixed number $\mathcal{N}_b$ of allowed bands. However, in some regions of the phase space, one can have more than one kind of solution of the saddle-point equation (3). In this situation, solutions with a different number of bands $\mathcal{N}_b^{(1)}$, $\mathcal{N}_b^{(2)}$, ... are present simultaneously. When such an overlap appears, one of the solutions, say $\mathcal{N}_b^{(k)}$, has the lowest free energy $\mathcal{F}_N^{(k)}$, and this solution is dominant, while the others are subdominant. Moreover, numerical calculations [12] showed that some special regimes exist in which the

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bulk spectral density obtained as a solution to the saddle-point equation (3) differs significantly from the genuine level density computed numerically within the framework of the orthogonal polynomial technique. It was then argued that such a genuine density of levels cannot be interpreted as a multiband solution with an integer number of bands. A full understanding of this phenomenon is still absent.

Recently, interest was renewed in multiband regimes in invariant random matrix ensembles. An analysis based on a loop equation technique [13,14] showed that fingerprints of phase transitions appear not only in the Dyson density but also in the (universal) wide-range eigenvalue correlators, which in the multiband phases differ from those known in the single-band phase [15–17]. A renormalization-group approach developed in Ref. [18] supported the results found in Refs. [13,14] for the particular case of two allowed bands, referring a new type of universal wide-range eigenlevel correlators to an additional attractive fixed point of a renormalization group transformation.

The method of loop equations [13,14], used for a treatment of non-Gaussian, unitary invariant, random matrix ensembles fallen in a multiband phase, is only suitable for computing the global characteristics of spectrum. Therefore, an appropriate approach is needed that is capable of analyzing local characteristics of the spectrum (manifested on the scale of a few eigenlevels). A possibility to probe the local properties of the eigenspectrum is offered by the method of orthogonal polynomials. A step in this direction was taken in a recent paper [19], where an ansatz was proposed for large-N asymptotes of orthogonal polynomials associated with a random matrix ensemble having two allowed bands in its spectrum. Because the asymptotic formula proposed there is of the Plancherel-Rotach type [20], it is only applicable for studying eigenvalue correlations in the spectrum bulk and cannot be used for studying local correlations in an arbitrary spectrum range (for example, near the edges of two-band eigenvalue support).

The aim of the present paper is to develop a new approach (within an orthogonal polynomial scheme) that allows for a unified treatment of eigenlevel correlations in the unitary invariant U(N) matrix model ($\beta=2$) with a forbidden gap. This is a further extension of the Shohat method [21,22] that has been used previously by the authors to study U(N) invariant ensembles of large random matrices in the single-band phase [23,24]. In particular, we are able to study both the fine structure of local characteristics of the spectrum in different scaling limits and smoothed global spectral correlations. Our treatment is based on the direct reconstruction of spectral correlations from the recurrence equation for the corresponding orthogonal polynomials.

II. GENERAL RELATIONS

In this section we briefly review the orthogonal polynomial technique [8]. The n-point correlation function, which describes the probability density to find n levels around each of the points $\varepsilon_1, \ldots, \varepsilon_n$ when the positions of the remaining levels are unobserved, is defined by the formula

$$R_n(\varepsilon_1, \ldots, \varepsilon_n) = \frac{N!}{(N-n)!}\int_{-\infty}^{+\infty} P(\{\varepsilon\}) \prod_{k=n+1}^{N} d\varepsilon_k. \quad (4)$$

This correlation function can explicitly be expressed in terms of the two-point kernel $K_N(\varepsilon,\varepsilon')$ as follows:

$$R_n(\varepsilon_1, \ldots, \varepsilon_n) = \det\|K_N(\varepsilon_i, \varepsilon_j)\|_{i,j=1,\ldots,n}. \quad (5)$$

Here,

$$K_N(\varepsilon,\varepsilon') = c_N \frac{\varphi_N(\varepsilon')\varphi_{N-1}(\varepsilon) - \varphi_N(\varepsilon)\varphi_{N-1}(\varepsilon')}{\varepsilon' - \varepsilon}, \quad (6)$$

and the ‘‘eigenfunctions’’

$$\varphi_n(\varepsilon) = P_n(\varepsilon)\ \exp\{-V(\varepsilon)\} \quad (7)$$

are determined by the set of polynomials orthogonal with respect to the measure $d\mu(\varepsilon) = \exp\{-2V(\varepsilon)\}d\varepsilon$,

$$\int_{-\infty}^{+\infty} d\mu(\varepsilon) P_n(\varepsilon) P_m(\varepsilon) = \delta_{nm}, \quad (8)$$

and obeying the recurrence equation

$$\varepsilon P_{n-1}(\varepsilon) = c_n P_n(\varepsilon) + c_{n-1} P_{n-2}(\varepsilon). \quad (9)$$

The recurrence coefficients c_n entering Eqs. (6) and (9) are uniquely determined by the measure $d\mu$. Equations (5) and (6) demonstrate that the problem of eigenvalue correlations is reduced to that of finding asymptotes for the eigenfunctions φ_N.

III. MAPPING RECURRENCE EQUATION ONTO DIFFERENTIAL EQUATION

To map a recurrence Eq. (9) onto a second-order differential equation for eigenfunctions φ_n, we note that the first derivative $dP_n/d\varepsilon$ can be represented as [21,22]

$$\frac{dP_n}{d\varepsilon} = A_n(\varepsilon) P_{n-1} - B_n(\varepsilon) P_n, \quad (10)$$

where

$$A_n(\varepsilon) = 2c_n \int d\mu(t) \frac{V'(t) - V'(\varepsilon)}{t-\varepsilon} P_n^2(t), \quad (11)$$

$$B_n(\varepsilon) = 2c_n \int d\mu(t) \frac{V'(t) - V'(\varepsilon)}{t-\varepsilon} P_n(t) P_{n-1}(t). \quad (12)$$

Then, by using Eqs. (9) and (10), one obtains after some algebra that the fictitious wave function φ_n given by Eq. (7) satisfies the following differential equation:

$$\frac{d^2\varphi_n(\varepsilon)}{d\varepsilon^2} - \mathcal{F}_n(\varepsilon)\frac{d\varphi_n(\varepsilon)}{d\varepsilon} + \mathcal{G}_n(\varepsilon)\varphi_n(\varepsilon) = 0. \quad (13)$$

Here,

$$\mathcal{F}_n(\varepsilon) = \frac{1}{A_n}\frac{dA_n}{d\varepsilon} \quad (14)$$

and

$$\mathcal{G}_n(\varepsilon)=\frac{dB_n}{d\varepsilon}+\frac{c_n}{c_{n-1}}A_nA_{n-1}-B_n\left(B_n+2\frac{dV}{d\varepsilon}+\frac{1}{A_n}\frac{dA_n}{d\varepsilon}\right)+\frac{d^2V}{d\varepsilon^2}-\left(\frac{dV}{d\varepsilon}\right)^2-\frac{1}{A_n}\frac{dA_n}{d\varepsilon}\frac{dV}{d\varepsilon}. \tag{15}$$

Equation (13) is valid for arbitrary n. We note that despite the generality of the differential equation obtained, its practical use is quite restricted since the functions $\mathcal{F}_n(\lambda)$ and $\mathcal{G}_n(\lambda)$ entering Eq. (13) can be calculated explicitly only for rather simple measures $d\mu$. Nevertheless, an asymptotic analysis of this equation is available in the limit $n=N\gg 1$, which is of great interest in random matrix theory.

A. Single-band phase

The single-band phase corresponds to monotonic confinement potentials or to those having light local extrema. Corresponding asymptotic analysis has been carried out by the authors in Refs. [23,24]. For further comparison with a two-band-phase solution, we give a differential equation for $\varphi_N^{(\mathrm{I})}(\varepsilon)$ obtained in the leading order in $N\gg 1$ [upper index indicates that the single-band phase is considered]:

$$\frac{d^2\varphi_N^{(\mathrm{I})}(\varepsilon)}{d\varepsilon^2}-\left[\frac{d}{d\varepsilon}\ln\left(\frac{\pi\nu_D^{(\mathrm{I})}(\varepsilon)}{\sqrt{\mathcal{D}_N^2-\varepsilon^2}}\right)\right]\frac{d\varphi_N^{(\mathrm{I})}(\varepsilon)}{d\varepsilon}+[\pi\nu_D^{(\mathrm{I})}(\varepsilon)]^2\varphi_N^{(\mathrm{I})}(\varepsilon)=0. \tag{16}$$

It is remarkable that Eq. (16) does not contain the confinement potential explicitly, but only involves the Dyson density

$$\nu_D^{(\mathrm{I})}(\varepsilon)=\frac{2}{\pi^2}\mathrm{P}\int_0^{\mathcal{D}_N}\frac{t\,dt}{t^2-\varepsilon^2}\frac{dV}{dt}\sqrt{\frac{1-\varepsilon^2/\mathcal{D}_N^2}{1-t^2/\mathcal{D}_N^2}} \tag{17}$$

corresponding to the single-band phase and analytically continued on the entire real axis; $\mathcal{D}_N$ is the soft edge of the spectrum, being the positive root of the integral equation

$$\int_0^{\mathcal{D}_N}\frac{dV}{dt}\frac{t\,dt}{\sqrt{\mathcal{D}_N^2-t^2}}=\frac{\pi N}{2}. \tag{18}$$

It has been shown that for a nonsingular confinement potential, solutions of Eq. (16) lead to the universal sine kernel in the bulk scaling limit, and to the so-called G-multicritical correlations in the soft-edge scaling limit [24]. An additional logarithmic singularity of confinement potential introduces additional terms into Eq. (16), giving rise to the universal Bessel correlations in the origin scaling limit [25,23]. For further progress in the field, see the very recent paper [26].

B. Two-band phase

Let us consider the situation when the confinement potential has two deep wells leading to the Dyson density supported on two disjoint intervals located symmetrically about the origin, $\mathcal{D}_N^-<|\varepsilon|<\mathcal{D}_N^+$. In this situation, the recurrence coefficients c_n entering Eq. (9) are known to be double-valued functions of the number n [1,10]. This means that for $n=N\gg 1$, one must distinguish between coefficients $c_{N\pm 2q}\approx c_N$ and coefficients $c_{N-1\pm 2q}\approx c_{N-1}$, belonging to two different smooth (in index) subsequences; here, integer $q\sim O(N^0)$. Bearing this in mind, the large-N version of recurrence equation (9) can be rewritten as

$$[\varepsilon^2-(c_N^2+c_{N-1}^2)]P_N(\varepsilon)=c_Nc_{N-1}[P_{N-1}(\varepsilon)+P_{N+1}(\varepsilon)], \tag{19}$$

from which we get the following asymptotic identities:

$$\varepsilon^{2\lambda}P_N(\varepsilon)=(c_N^2+c_{N-1}^2)^\lambda\sum_{k=0}^{\lambda}\binom{\lambda}{k}\left(\frac{c_Nc_{N-1}}{c_N^2+c_{N-1}^2}\right)^k\sum_{j=0}^{k}\binom{k}{j}\times P_{N+4j-2k}(\varepsilon) \tag{20}$$

and

$$\varepsilon^{2\lambda+1}P_N(\varepsilon)=(c_N^2+c_{N-1}^2)^\lambda\sum_{k=0}^{\lambda}\binom{\lambda}{k}\left(\frac{c_Nc_{N-1}}{c_N^2+c_{N-1}^2}\right)^k\sum_{j=0}^{k}\binom{k}{j}\times[c_{N-1}P_{N+4j-2k+1}(\varepsilon)+c_NP_{N+4j-2k-1}(\varepsilon)] \tag{21}$$

with integer $\lambda\geq 0$.

Expansions Eqs. (20) and (21) make it possible to compute the required functions $\mathcal{F}_N$ and $\mathcal{G}_N$ entering the differential equation (13) for fictitious wave functions in the limit $N\gg 1$. Substituting the explicit form of the confinement potential set by Eq. (2) into Eqs. (11) and (12), we obtain

$$A_N(\varepsilon)=2c_N\sum_{k=1}^{p}d_k\sum_{\lambda=1}^{2k-1}\varepsilon^{\lambda-1}\int d\mu(t)P_N^2(t)t^{2k-\lambda-1} \tag{22}$$

and

$$B_N(\varepsilon)=2c_N\sum_{k=1}^{p}d_k\sum_{\lambda=1}^{2k-1}\varepsilon^{\lambda-1}\times\int d\mu(t)P_N(t)P_{N-1}(t)t^{2k-\lambda-1}, \tag{23}$$

respectively. Both integrals above can be calculated using expansions Eqs. (20), (21), and exploiting the orthogonality expressed by Eq. (8). Detailed calculations, given in Appendixes A and B, lead to the following results:

$$A_N(\varepsilon)=\frac{2}{\pi}[\mathcal{D}_N^+-(-1)^N\mathcal{D}_N^-]\mathrm{P}\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+}\frac{dV}{dt}\frac{dt}{t^2-\varepsilon^2}\times\frac{t^2}{\sqrt{[(\mathcal{D}_N^+)^2-t^2][t^2-(\mathcal{D}_N^-)^2]}}, \tag{24}$$

$$B_N(\varepsilon)=\frac{2}{\pi}\varepsilon\mathrm{P}\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+}\frac{dV}{dt}\frac{t^2-(-1)^N\mathcal{D}_N^-\mathcal{D}_N^+}{\sqrt{[(\mathcal{D}_N^+)^2-t^2][t^2-(\mathcal{D}_N^-)^2]}}\frac{dt}{t^2-\varepsilon^2}-\frac{dV}{d\varepsilon}. \tag{25}$$

Having obtained the explicit expressions for functions A_N and B_N, it is easy to verify that coefficients $\mathcal{F}_n(\varepsilon)$ and $\mathcal{G}_n(\varepsilon)$ entering the differential equation (13) for the fictitious wave function $\varphi_n^{(\mathrm{II})}(\varepsilon)$ may be expressed in terms of the Dyson density $\nu_D^{(\mathrm{II})}$ in the two-cut phase supported on two disconnected intervals $\varepsilon \in (-\mathcal{D}_N^+, -\mathcal{D}_N^-) \cup (\mathcal{D}_N^-, \mathcal{D}_N^+)$,

$$\nu_D^{(\mathrm{II})}(\varepsilon) = \frac{2}{\pi^2}|\varepsilon| \sqrt{[(\mathcal{D}_N^+)^2-\varepsilon^2][\varepsilon^2-(\mathcal{D}_N^-)^2]} \times \mathrm{P}\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+} dt \frac{dV/dt}{t^2-\varepsilon^2} \frac{1}{\sqrt{[(\mathcal{D}_N^+)^2-t^2][t^2-(\mathcal{D}_N^-)^2]}} \tag{26}$$

when $N \gg 1$. Namely, Eqs. (14), (15), (24), and (25) yield

$$\mathcal{F}_N(\varepsilon) = \frac{d}{d\varepsilon} \ln\left(\frac{\pi|\varepsilon|\nu_D^{(\mathrm{II})}(\varepsilon)}{\sqrt{[(\mathcal{D}_N^+)^2-\varepsilon^2][\varepsilon^2-(\mathcal{D}_N^-)^2]}} \right), \tag{27}$$

$$\mathcal{G}_N(\varepsilon) = [\pi \nu_D^{(\mathrm{II})}(\varepsilon)]^2 + \frac{\pi\nu_D^{(\mathrm{II})}(\varepsilon)}{|\varepsilon|\sqrt{[(\mathcal{D}_N^+)^2-\varepsilon^2][\varepsilon^2-(\mathcal{D}_N^-)^2]}} \times [\varepsilon^2 + (-1)^N \mathcal{D}_N^- \mathcal{D}_N^+]. \tag{28}$$

In the large-N limit, the second term in Eq. (28) can be neglected provided ε belongs to the one of allowed bands, so that $\varphi_N^{(\mathrm{II})}(\varepsilon)$ satisfies the following asymptotic differential equation in the two-cut phase:

$$\frac{d^2\varphi_N^{(\mathrm{II})}(\varepsilon)}{d\varepsilon^2} - \left[\frac{d}{d\varepsilon} \ln\left(\frac{\pi|\varepsilon|\nu_D^{(\mathrm{II})}(\varepsilon)}{\sqrt{[(\mathcal{D}_N^+)^2-\varepsilon^2][\varepsilon^2-(\mathcal{D}_N^-)^2]}} \right) \right] \frac{d\varphi_N^{(\mathrm{II})}(\varepsilon)}{d\varepsilon} + [\pi \nu_D^{(\mathrm{II})}(\varepsilon)]^2 \varphi_N^{(\mathrm{II})}(\varepsilon) = 0. \tag{29}$$

We recall that $\mathcal{D}_N^-$ and $\mathcal{D}_N^+$ are the end points of the eigenvalue support that obey the two integral equations

$$\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+} \frac{dV}{dt} \frac{t^2 dt}{\sqrt{[(\mathcal{D}_N^+)^2-t^2][t^2-(\mathcal{D}_N^-)^2]}} = \frac{\pi N}{2}, \tag{30}$$

$$\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+} \frac{dV}{dt} \frac{dt}{\sqrt{[(\mathcal{D}_N^+)^2-t^2][t^2-(\mathcal{D}_N^-)^2]}} = 0, \tag{31}$$

obtained in Appendix C. One can verify that as $\mathcal{D}_N^-$ tends to zero, we recover Eq. (16) valid in the single-band regime.

IV. LOCAL EIGENVALUE CORRELATIONS

Eigenvalue correlations in the spectra of two-band random matrices are completely determined by the Dyson density of states entering the effective Schrödinger equation (29).

(i) In the spectrum bulk, the Dyson density is a well-behaved function that can be taken approximately as a constant on the scale of a few eigenlevels. Then, in the vicinity of some ε_0 that is chosen to be far enough from the spectrum end points $\pm \mathcal{D}_N^\pm$, Eq. (29) takes the form

$$\frac{d^2\varphi_N^{(\mathrm{II})}(\varepsilon)}{d\varepsilon^2} + [\pi/\Delta(\varepsilon_0)]^2 \varphi_N^{(\mathrm{II})}(\varepsilon) = 0, \tag{32}$$

with $\Delta(\varepsilon_0) = 1/\nu_D^{(\mathrm{II})}(\varepsilon_0)$ being the mean level spacing in the vicinity of ε_0. Clearly, the universal sine law for the two-point kernel, Eq. (6), follows immediately.

(ii) Eigenvalue correlations near the end points of an eigenvalue support are determined by the Dyson density as well. Noting that in the vicinity of $|\varepsilon| = \mathcal{D}_N^\pm$ the Dyson density can be represented in the form [27,24]

$$\nu_D^{(\mathrm{II})}(\varepsilon) = \left[\pm \left(1 - \frac{\varepsilon^2}{(\mathcal{D}_N^\pm)^2} \right) \right]^{m+1/2} \mathcal{R}_N\left(\frac{\varepsilon}{\mathcal{D}_N^\pm} \right), \tag{33}$$

where $\mathcal{R}_N(\pm 1) \neq 0$ and m is the order of multicriticality, we readily recover the universal multicritical correlations previously found [24] in the soft-edge scaling limit for the U(N) invariant matrix model in the single-band phase.

V. SMOOTHED CONNECTED "DENSITY-DENSITY" CORRELATOR

Let us turn to the study of the connected "density-density" correlator that is expressed in terms of the two-point kernel, Eq. (6), as follows:

$$\langle \delta\nu_N(\varepsilon)\delta\nu_N(\varepsilon')\rangle_{\mathrm{II}} = -\frac{c_N^2}{(\varepsilon-\varepsilon')^2} \{ \varphi_N^2(\varepsilon)\varphi_{N-1}^2(\varepsilon') + \varphi_N^2(\varepsilon')\varphi_{N-1}^2(\varepsilon) - 2\varphi_N(\varepsilon)\varphi_{N-1}(\varepsilon)\varphi_N(\varepsilon')\varphi_{N-1}(\varepsilon') \}, \tag{34}$$

where $\varepsilon \neq \varepsilon'$, and the upper index (II) in φ_N has been omitted for brevity. We still deal with the two-band phase. Equation (34) contains rapid oscillations on the scale of the mean level spacing. These oscillations are due to the presence in Eq. (34) of oscillating wave functions φ_N and φ_{N-1}.

To average over the rapid oscillations, we integrate, over the entire real axis, rapidly varying wave functions in Eq. (34) multiplied by an arbitrary, smooth, slowly varying function. To illustrate the idea, consider the integral

$$I_{\mathsf{f}} = \int_{-\infty}^{+\infty} d\varepsilon\, \varphi_N^2(\varepsilon) \mathsf{f}(\varepsilon), \tag{35}$$

where $\mathsf{f}(\varepsilon)$ is an arbitrary slowly varying function that should be chosen to be even due to the evenness of $\varphi_N^2(\varepsilon)$. Setting

$$\mathsf{f}(\varepsilon) = \sum_{\alpha=0}^{\infty} \mathsf{f}_\alpha \varepsilon^{2\alpha}, \tag{36}$$

we immediately obtain with the help of Eqs. (A1) and (A6) that

$$I_{\mathsf{f}}=\sum_{\alpha=0}^{\infty} \mathsf{f}_{\alpha}\Lambda_{2\alpha}=\frac{2}{\pi}\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+}\frac{\varepsilon \mathsf{f}(\varepsilon)d\varepsilon}{\sqrt{[(\mathcal{D}_N^+)^2-\varepsilon^2][\varepsilon^2-(\mathcal{D}_N^-)^2]}}. \tag{37}$$

Bearing in mind that both $\mathsf{f}(\varepsilon)$ and $\varphi_N^2(\varepsilon)$ are even functions, the last integral can be transformed as follows:

$$I_{\mathsf{f}}=\int_{-\infty}^{+\infty}d\varepsilon\, \varphi_N^2(\varepsilon)\mathsf{f}(\varepsilon)$$
$$=\frac{1}{\pi}\int_{\mathcal{D}_N^-<|\varepsilon|<\mathcal{D}_N^+}\frac{|\varepsilon|\mathsf{f}(\varepsilon)d\varepsilon}{\sqrt{[(\mathcal{D}_N^+)^2-\varepsilon^2][\varepsilon^2-(\mathcal{D}_N^-)^2]}}, \tag{38}$$

from which we conclude that in the large-N limit,

$$\overline{\varphi_N^2(\varepsilon)}=\frac{1}{\pi}\frac{|\varepsilon|}{\sqrt{[(\mathcal{D}_N^+)^2-\varepsilon^2][\varepsilon^2-(\mathcal{D}_N^-)^2]}}$$
$$\times\Theta(\mathcal{D}_N^+-|\varepsilon|)\Theta(|\varepsilon|-\mathcal{D}_N^-). \tag{39}$$

The same procedure should be carried out with expression $\varphi_N(\varepsilon)\varphi_{N-1}(\varepsilon)$ in Eq. (34). Since this construction is an odd function of ε, we have to consider the integral

$$I_{\mathsf{g}}=\int_{-\infty}^{+\infty}d\varepsilon\, \varphi_N(\varepsilon)\varphi_{N-1}(\varepsilon)\mathsf{g}(\varepsilon), \tag{40}$$

with

$$\mathsf{g}(\varepsilon)=\sum_{\alpha=0}^{\infty}\mathsf{g}_{\alpha}\varepsilon^{2\alpha+1} \tag{41}$$

being a smooth odd function. It is easy to see with the help of Eqs. (B1), (B5), and (C7) that

$$I_{\mathsf{g}}=\sum_{\alpha=0}^{\infty}\mathsf{g}_{\alpha}\Gamma_{2\alpha+1}$$
$$=\frac{2}{\pi[\mathcal{D}_N^+-(-1)^N\mathcal{D}_N^-]}\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+}\frac{\mathsf{g}(\varepsilon)[\varepsilon^2-(-1)^N\mathcal{D}_N^-\mathcal{D}_N^+]d\varepsilon}{\sqrt{[(\mathcal{D}_N^+)^2-\varepsilon^2][\varepsilon^2-(\mathcal{D}_N^-)^2]}}. \tag{42}$$

Exploiting the oddness of $\mathsf{g}(\varepsilon)$ and $\varphi_N(\varepsilon)\varphi_{N-1}(\varepsilon)$, we write Eq. (42) in the form

$$I_{\mathsf{g}}=\int_{-\infty}^{+\infty}d\,\varepsilon\varphi_N(\varepsilon)\varphi_{N-1}(\varepsilon)\mathsf{g}(\varepsilon)$$
$$=\frac{1}{\pi[\mathcal{D}_N^+-(-1)^N\mathcal{D}_N^-]}\int_{\mathcal{D}_N^-<|\varepsilon|<\mathcal{D}_N^+}d\varepsilon$$
$$\times\frac{\mathsf{g}(\varepsilon)[\varepsilon^2-(-1)^N\mathcal{D}_N^-\mathcal{D}_N^+]\mathrm{sgn}(\varepsilon)}{\sqrt{[(\mathcal{D}_N^+)^2-\varepsilon^2][\varepsilon^2-(\mathcal{D}_N^-)^2]}}. \tag{43}$$

Equation (43) leads us to the conclusion that in the large-N limit,

$$\overline{\varphi_N(\varepsilon)\varphi_{N-1}(\varepsilon)}$$
$$=\frac{\mathrm{sgn}(\varepsilon)}{\pi[\mathcal{D}_N^+-(-1)^N\mathcal{D}_N^-]}\frac{\varepsilon^2-(-1)^N\mathcal{D}_N^-\mathcal{D}_N^+}{\sqrt{[(\mathcal{D}_N^+)^2-\varepsilon^2][\varepsilon^2-(\mathcal{D}_N^-)^2]}}$$
$$\times\Theta(\mathcal{D}_N^+-|\varepsilon|)\Theta(|\varepsilon|-\mathcal{D}_N^-). \tag{44}$$

Combining Eqs. (34), (39), (44), and (C7), we finally arrive at the following formula for the smoothed "density-density" correlator:

$$\overline{\langle\delta\nu_N(\varepsilon)\delta\nu_N(\varepsilon')\rangle}_{\mathrm{II}}=-\frac{\mathrm{sgn}(\varepsilon\varepsilon')}{2\pi^2}\Theta(\mathcal{D}_N^+-|\varepsilon|)\Theta(|\varepsilon|-\mathcal{D}_N^-)\Theta(\mathcal{D}_N^+-|\varepsilon'|)\Theta(|\varepsilon'|-\mathcal{D}_N^-)$$
$$\times\left\{\frac{1}{(\varepsilon-\varepsilon')^2}\frac{[\varepsilon\varepsilon'-(\mathcal{D}_N^-)^2][(\mathcal{D}_N^+)^2-\varepsilon\varepsilon']}{\sqrt{[(\mathcal{D}_N^+)^2-\varepsilon^2][\varepsilon^2-(\mathcal{D}_N^-)^2]}\sqrt{[(\mathcal{D}_N^+)^2-\varepsilon'^2][\varepsilon'^2-(\mathcal{D}_N^-)^2]}}\right.$$
$$\left.+(-1)^N\frac{\mathcal{D}_N^-\mathcal{D}_N^+}{\sqrt{[(\mathcal{D}_N^+)^2-\varepsilon^2][\varepsilon^2-(\mathcal{D}_N^-)^2]}\sqrt{[(\mathcal{D}_N^+)^2-\varepsilon'^2][\varepsilon'^2-(\mathcal{D}_N^-)^2]}}\right\}. \tag{45}$$

The same formula can be obtained by WKB by solving Eq. (29), using definition Eq. (34) followed by averaging over rapid oscillations. It can be verified that for N even, this result coincides with Eq. (6.6) of Ref. [19], where it was obtained by a completely different method, and for the case of odd N being omitted.

It is seen from Eq. (45) that the smoothed "density-density" correlator in the two-band phase is a new universal function in random matrix theory. It is universal in the sense that the information of the distribution Eq. (1) is encoded into the "density-density" correlator only through the end points $\mathcal{D}_N^{\pm}$ of the eigenvalue support. A striking feature of the new universal function Eq. (45) is its *sharp* dependence on the oddness or evenness of the dimension N of the random matrices whose spectra are *bounded*. The origin of this unusual large-N behavior will be discussed in the next section.

Finally, let us speculate about the universal correlator Eq. (45) in the limit of *unbounded* spectrum, $\mathcal{D}_N^+ \to \infty$, with a gap. Inasmuch as it describes correlations between the eigenlevels that are repelled from each other in accordance with the logarithmic law, which is known to be realized [28,29] in the weakly disordered systems on the energy scale $|\varepsilon - \varepsilon'| \ll E_c$ (E_c is the Thouless energy), we may *conjecture* that the corresponding limiting universal expression

$$\lim_{\mathcal{D}_N^+ \to +\infty} \overline{\langle \delta\nu_N(\varepsilon)\delta\nu_N(\varepsilon')\rangle}_{\mathrm{II}} = -\frac{\mathrm{sgn}(\varepsilon\varepsilon')}{2\pi^2(\varepsilon-\varepsilon')^2}\frac{\varepsilon\varepsilon'-\Delta^2}{\sqrt{[\varepsilon^2-\Delta^2][\varepsilon'^2-\Delta^2]}} \times \Theta(|\varepsilon|-\Delta)\Theta(|\varepsilon'|-\Delta) \tag{46}$$

reflects the universal properties of real chaotic systems with a forbidden gap $\Delta = \mathcal{D}_N^-$ and broken time reversal symmetry, provided $|\varepsilon - \varepsilon'| \ll E_c$. In two limiting situations (i) of gapless spectrum, $\Delta = 0$, and (ii) far from the gap, $|\varepsilon|, |\varepsilon'| \gg \Delta$, the correlator Eq. (46) coincides with that known in the random matrix theory of gapless ensembles [16,17] and derived in Ref. [28] within the framework of a diagrammatic technique for the spectrum of an electron in a random impurity potential.

VI. CONCLUDING REMARKS

In this study we developed a unified formalism allowing the computation of both global and local spectral characteristics of U(N) invariant ensembles of large random matrices possessing $Z2$ symmetry, and deformed in such a way that their spectra contain a forbidden gap. We proved that in the pure two-band phase, the local eigenvalue correlations are insensitive to this deformation both in the bulk and soft-edge scaling limits. In contrast, global smoothed eigenvalue correlations in the two-band phase differ drastically from those in the single-band phase, and generically satisfy a new universal law, Eq. (45), which is unusually sensitive to the oddness or evenness of the random matrix dimension if the spectrum support is *bounded*. On the formal level, this sensitivity is a direct consequence of the "period-two" behavior [1,10] of the recurrence coefficients c_n [see Eq. (9)] that is characteristic of the two-band phase of the reduced Hermitian matrix model. To see this, consider the simplest connected correlator $\langle \mathrm{Tr}\mathbf{H}\mathrm{Tr}\mathbf{H}\rangle_c$ that can be *exactly* represented in terms of recurrence coefficients for any n,

$$\langle \mathrm{Tr}\mathbf{H}\mathrm{Tr}\mathbf{H}\rangle_c = c_n^2. \tag{47}$$

Since in the two-band phase c_n is a double-valued function of index n, alternating between two different functions as n goes from odd to even, the large-N limit of the correlator $\langle \mathrm{Tr}\mathbf{H}\mathrm{Tr}\mathbf{H}\rangle_c$ strongly depends on whether infinity is approached through odd or even N. Then, an implementation of a double-valued behavior of c_n into the higher order correlators of the form $\langle \mathrm{Tr}\mathbf{H}^k\mathrm{Tr}\mathbf{H}^l\rangle_c$ contributing to the connected "density-density" correlator gives rise to the new universal expression Eq. (45).

Let us, however, point out that no such sensitivity has been detected in a number of previous studies [13,14] exploiting a loop-equation technique. This is due to the following reasons. In the method of loop equations, used for a treatment of non-Gaussian random matrix ensembles fallen in a multiband phase, one is forced to keep the most general (nonsymmetric) confinement potential $V(\varepsilon) = \Sigma_{k=1}^{2p}\tilde{d}_k\varepsilon^k/k$ until the very end of the calculations, leading to a necessity to take the thermodynamic limit $N\to\infty$ prior to any others. Therefore, $Z2$ symmetry in this calculational scheme can only be implemented by restoring $Z2$ symmetry at the final stage of the calculations, setting all the extra coupling constants $\tilde{d}_{2k+1}$ to zero. Doing so, one arrives at the results reported in Refs. [13,14].

From this point of view, the formalism developed in this paper corresponds to the *opposite sequence* of thermodynamic and $Z2$-symmetry limits, since we have considered the random matrix model that possesses $Z2$ symmetry from the beginning. Qualitatively different large-N behavior of the smoothed connected "density-density" correlator, Eq. (45), and of the smoothed connected two-point Green's function given by Eq. (15) of Ref. [14] provides direct evidence that the order of thermodynamic and $Z2$-symmetry limits is indeed important when studying global spectral characteristics of multiband random matrices.

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APPENDIX A: CALCULATION OF THE FUNCTION $A_N(\varepsilon)$

Let us consider an integral

$$\Lambda_{2\sigma} = \int d\mu(t) P_N^2(t) t^{2\sigma} \tag{A1}$$

with integer $\sigma \geq 0$. Making use of Eq. (20), we rewrite $\Lambda_{2\sigma}$ in the form

$$\Lambda_{2\sigma} = (c_N^2 + c_{N-1}^2)^\sigma \sum_{k=0}^{\sigma} \binom{\sigma}{k} \times \left(\frac{c_N c_{N-1}}{c_N^2 + c_{N-1}^2}\right)^k \sum_{j=0}^{k} \binom{k}{j} \int d\mu(t) P_N(t) P_{N+4j-2k}(t). \tag{A2}$$

Orthogonality of the P_n allows us to integrate over the measure $d\mu$, thus simplifying Eq. (A2):

$$\Lambda_{2\sigma} = (c_N^2 + c_{N-1}^2)^\sigma \sum_{k=0}^{\sigma} \binom{\sigma}{k} \left(\frac{c_N c_{N-1}}{c_N^2 + c_{N-1}^2}\right)^k \sum_{j=0}^{k} \binom{k}{j} \delta_{2j}^k, \tag{A3}$$

where $\delta_{k'}^k$ is the Kronecker symbol. Using the integral representation

$$\delta_{k'}^{k} = \mathrm{Re}\int_0^{2\pi}\frac{d\theta}{2\pi}\exp\{i(k-k')\theta\}, \tag{A4}$$

one can perform the double summation in Eq. (A3):

$$\Lambda_{2\sigma} = \int_0^{2\pi}\frac{d\theta}{2\pi}(c_N^2+c_{N-1}^2+2c_N c_{N-1}\cos\theta)^{\sigma}. \tag{A5}$$

Introducing a new integration variable $t^2=c_N^2+c_{N-1}^2+2c_N c_{N-1}\cos\theta$, we derive an integral formula

$$\Lambda_{2\sigma} = \frac{2}{\pi}\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+}\frac{t^{2\sigma+1}dt}{\sqrt{[(\mathcal{D}_N^+)^2-t^2][t^2-(\mathcal{D}_N^-)^2]}} \tag{A6}$$

with

$$\mathcal{D}_N^{\pm} = |c_N \pm c_{N-1}|. \tag{A7}$$

Now, taking into account representation Eq. (A6) for $\Lambda_{2\sigma}$, and using the fact that $\Lambda_{2\sigma+1}\equiv 0$, we obtain from Eq. (22)

$$A_N(\varepsilon) = 2c_N\sum_{k=1}^{p} d_k\sum_{\sigma=1}^{k}\Lambda_{2(k-\sigma)}\varepsilon^{2\sigma-2}. \tag{A8}$$

Summing over σ yields

$$A_N(\varepsilon) = \frac{4c_N}{\pi}\sum_{k=1}^{p} d_k\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+}dt \times\frac{t}{\sqrt{[(\mathcal{D}_N^+)^2-t^2][t^2-(\mathcal{D}_N^-)^2]}}\,\frac{t^{2k}-\varepsilon^{2k}}{t^2-\varepsilon^2}, \tag{A9}$$

from which we get, with the help of Eq. (2),

$$A_N(\varepsilon) = \frac{4c_N}{\pi}\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+}\frac{dt}{\sqrt{[(\mathcal{D}_N^+)^2-t^2][t^2-(\mathcal{D}_N^-)^2]}}\,\frac{t}{t^2-\varepsilon^2} \times\left(t\frac{dV}{dt}-\varepsilon\frac{dV}{d\varepsilon}\right). \tag{A10}$$

Further, noting that

$$\mathrm{P}\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+}\frac{dt}{\sqrt{[(\mathcal{D}_N^+)^2-t^2][t^2-(\mathcal{D}_N^-)^2]}}\,\frac{t}{t^2-\varepsilon^2}\equiv 0, \tag{A11}$$

and taking into account Eq. (C7), leads to the final expression given by Eq. (24).

APPENDIX B: CALCULATION OF THE FUNCTION $B_N(\varepsilon)$

Let us consider an integral

$$\Gamma_{2\sigma+1} = \int d\mu(t)P_N(t)P_{N-1}(t)t^{2\sigma+1} \tag{B1}$$

with integer $\sigma\geqslant 0$. Making use of expansion Eq. (21), we rewrite Eq. (B1) in the form that allows us to perform the integration over the measure $d\mu$:

$$\Gamma_{2\sigma+1} = \frac{1}{2}(c_N^2+c_{N-1}^2)^{\sigma}\int d\mu(t)P_{N-1}(t)\sum_{k=0}^{\sigma}\binom{\sigma}{k} \times\left(\frac{c_N c_{N-1}}{c_N^2+c_{N-1}^2}\right)^{k}\sum_{j=0}^{k}\binom{k}{j}[c_{N-1}P_{N+4j-2k+1}(t) +c_N P_{N+4j-2k-1}(t)]. \tag{B2}$$

After integration, we get

$$\Gamma_{2\sigma+1} = \frac{1}{2}(c_N^2+c_{N-1}^2)^{\sigma}\sum_{k=0}^{\sigma}\binom{\sigma}{k}\left(\frac{c_N c_{N-1}}{c_N^2+c_{N-1}^2}\right)^{k}\sum_{j=0}^{k}\binom{k}{j} \times[c_{N-1}\delta_{2j+1}^{k}+c_N\delta_{2j}^{k}]. \tag{B3}$$

The double summation in Eq. (B3) can be performed using the integral representation for the Kronecker symbol given by Eq. (A4):

$$\Gamma_{2\sigma+1} = \frac{1}{2}\int_0^{2\pi}\frac{d\theta}{2\pi}(c_N^2+c_{N-1}^2+2c_N c_{N-1}\cos\theta)^{\sigma} \times[c_N+c_{N-1}\cos\theta]. \tag{B4}$$

Introducing a new integration variable $t^2=c_N^2+c_{N-1}^2+2c_N c_{N-1}\cos\theta$, we get

$$\Gamma_{2\sigma+1} = \frac{1}{\pi c_N}\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+}\frac{t^{2\sigma+1}dt}{\sqrt{[(\mathcal{D}_N^+)^2-t^2][t^2-(\mathcal{D}_N^-)^2]}} \times[t^2+c_N^2-c_{N-1}^2]. \tag{B5}$$

Then, Eqs. (23), (B1), and (B5) yield

$$B_N(\varepsilon) = \frac{2}{\pi}\sum_{k=1}^{p} d_k\sum_{\sigma=1}^{k-1}\Gamma_{2k-2\sigma-1}\varepsilon^{2\sigma-1}. \tag{B6}$$

Summing over σ leads to the integral expression

$$B_N(\varepsilon) = \frac{2}{\pi}\sum_{k=1}^{p} d_k\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+}dt\,\frac{[t^2+c_N^2-c_{N-1}^2]}{\sqrt{[(\mathcal{D}_N^+)^2-t^2][t^2-(\mathcal{D}_N^-)^2]}} \times\frac{\varepsilon t^{2k-1}-t\varepsilon^{2k-1}}{t^2-\varepsilon^2}$$
$$= \frac{2}{\pi}\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+}\frac{dt}{\sqrt{[(\mathcal{D}_N^+)^2-t^2][t^2-(\mathcal{D}_N^-)^2]}} \times\frac{t^2+c_N^2-c_{N-1}^2}{t^2-\varepsilon^2}\left(\varepsilon\frac{dV}{dt}-t\frac{dV}{d\varepsilon}\right). \tag{B7}$$

Now, taking into account Eqs. (A11) and (C6), we obtain Eq. (25).

APPENDIX C: SOFT EDGES OF EIGENVALUE SUPPORT

To find the equations determining the points $\mathcal{D}_N^{\pm}$ where the Dyson spectral density goes to zero, we start with the following formula from the theory of orthogonal polynomials [30]:

$$n=2c_n\int d\mu(t)\frac{dV}{dt}P_n(t)P_{n-1}(t). \tag{C1}$$

Let us use expansion Eq. (21) to calculate asymptotically the integral entering Eq. (C1) in the limit $n=N\gg 1$. It is easy to see that

$$\begin{aligned}N&=2c_N\sum_{\lambda=1}^{p}d_\lambda\int d\mu(t)P_N(t)P_{N-1}(t)t^{2\lambda-1}\\&=2c_N\sum_{\lambda=1}^{p}d_\lambda\Gamma_{2\lambda-1},\end{aligned} \tag{C2}$$

where $\Gamma_{2\lambda-1}$ is given by Eq. (B5). Then, we immediately obtain the relationship

$$N=\frac{2}{\pi}\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+}\frac{dt}{\sqrt{[(\mathcal{D}_N^+)^2-t^2][t^2-(\mathcal{D}_N^-)^2]}}\frac{dV}{dt}[t^2+c_N^2-c_{N-1}^2]. \tag{C3}$$

This result, rewritten for $n=N-1$, yields in the large-N limit

$$N=\frac{2}{\pi}\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+}\frac{dt}{\sqrt{[(\mathcal{D}_N^+)^2-t^2][t^2-(\mathcal{D}_N^-)^2]}}\frac{dV}{dt}[t^2+c_{N-1}^2-c_N^2]. \tag{C4}$$

Equations (C3) and (C4) yield two equations whose solutions determine the edge points $\mathcal{D}_N^\pm$:

$$\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+}\frac{t^2dt}{\sqrt{[(\mathcal{D}_N^+)^2-t^2][t^2-(\mathcal{D}_N^-)^2]}}\frac{dV}{dt}=\frac{\pi N}{2} \tag{C5}$$

and

$$\int_{\mathcal{D}_N^-}^{\mathcal{D}_N^+}\frac{dt}{\sqrt{[(\mathcal{D}_N^+)^2-t^2][t^2-(\mathcal{D}_N^-)^2]}}\frac{dV}{dt}=0. \tag{C6}$$

Finally, we note that because $P_{-1}(\varepsilon)=0$, it follows from Eq. (9) that $c_0=0$, and as a consequence, an even branch c_{2N} always lies lower than an odd branch $c_{2N\pm1}$, so that $c_{2N}<c_{2N\pm1}$. Then, we may conclude from Eq. (A7) that

$$c_N=\frac{\mathcal{D}_N^+-(-1)^N\mathcal{D}_N^-}{2}. \tag{C7}$$

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